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ROTATION OF TRAJECTORIES OF LIPSCHITZ VECTOR FIELDS

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Abstract. – We prove that in finite time a trajectory of a Lipschitz vector field in $\mathbb{R}^n$ can not have infinite rotation around a given point. This result extends to the mutual rotation of two trajectories of a field in $\mathbb{R}^3$: this rotation is bounded from above on any finite time interval. The bounds we give are only in terms of the Lipschitz constant of the field and the length of the time interval.

1. Introduction

In this paper we investigate the tameness of a geometric behavior of trajectories of vector fields: the rotation of such a trajectory around a point or the mutual rotation of two trajectories.

Of course a lot of results have been published on the geometry of solutions of differential equations and trajectories of vector fields, and we simply cannot give an extensive review or even a bibliography on the subject. Nevertheless, we have at our disposal only few global theories and general results concerning the tameness of trajectories. In this very short introduction we just want to focus on two of them.

In Gabrielov-Hovanskii's theory of Pfaffian sets, a Pfaffian function f on an open set $\mathcal{U} \subset \mathbb{R}^n$, is defined as a function that can be written in the following way: $f(x) = P(x, c_1(x), \cdots, c_m(x))$, where P is a polynomial and the c_i's are analytic solutions of the polynomial triangular differential system:

$$Dc_i(x) = \sum_{j=1}^{m} P_{i,j}(x, c_1(x), \cdots, c_i(x))\ dx_j, \quad i \in \{1, \cdots, m\}. \qquad (*)$$

Then if we consider the Pfaffian structure, that is to say the smallest structure containing the semi-Pfaffian sets, it has been proved in [Wi], using a Bezout type theorem of Hovanskii

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([Ho 1], [Ho 2]), that this structure is o-minimal (see [Co], [Dr], [Dr-Mi], [Sh]): for short the number of connected components of sets in such a structure is finite, and consequently no Pfaffian curve (nor set) may infinitely oscillate or spiral around a point and two such curves have bounded mutual rotation.

This theory presents a large class of tame objects coming from differential equations, but when we deal with vector fields, the only way to a priori be sure that the trajectories belong to this category is to assume that the trajectory is a Pfaffian function itself, satisfying equation ($*$). We deduce from this assumption that the field depends only on one variable. A too restrictive hypothesis that obviously not allows a wild behavior for trajectories.

On the other hand, starting with a given vector field and studying the local geometry of its trajectories in a neighborhood of one of its singularities, we know that this geometry is tame, following [Ku-Mo] and [Ku-Mo-Pa], provided that the field is an analytic gradient vector field: the rotation of a trajectory of an analytic gradient vector field around one of its singularities is finite. As a consequence the limit of the secant lines to the trajectory exists (Thom's Gradient Conjecture). Let us notice that we do not know wether a trajectory of an analytic gradient vector field lies in some o-minimal structure, although we know that trajectories of the gradient of a function definable in a given o-minimal structure have finite length (see [Ku]). On the local behavior of trajectories we would like to refer to the number of deep results produced by the *Spanish-Dijon School*, and specially the most recent ones: [Ca-Mo-Ro], [Ca-Mo-Sa 1,2,3], [Bl-Mo-Ro].

The aim of this paper is to give general results about rotation of trajectories of a vector field, with no restrictive assumptions on the nature of the field, besides the Lipschitz property which is a minimal hypothesis for the existence of trajectories.

Of course, in this direction, we cannot hope to treat infinite time phenomena, as it is done in [Ku-Mo-Pa] and the Pfaffian theory, because it is well-known that complete trajectories of polynomial vector fields in the plane may spiral around a singularity of the field (see for instance Remark of Section 3.1). We thus obtain our bounds for trajectories defined on a finite time interval. For bounds obtained in the same spirit, the reader may report to [Gr-Yo], [Ho-Ya], [No-Ya 1,2], [Ya].

The paper is organized as follows.

In Section 2 we introduce and compare some notions of absolute and topological rotation for trajectories around an affine subspace in $\mathbb{R}^n$ or for two trajectories around each others in $\mathbb{R}^3$.

In Section 3 we first notice that the rotation of any trajectory of a Lipschitz vector field around its stationary points is bounded in terms of the Lipschitz constant (and the time elapsed) only (Proposition 3.1). The same is true for the rotation of any trajectory around a linear invariant subspace of the field (Proposition 3.2). Moreover, while the rotation velocity of a trajectory around a *non-stationary point* of the field may tend to infinity, we prove our first main result: the "total" rotation around such a point is still bounded in terms of the Lipschitz constant and the time interval (Theorem 3.4). Our second main result is a consequent of the first one: we give a uniform bound for the rotation of any two trajectories of a given Lipschitz vector field, in terms of the time interval and the Lipschitz constant (Theorem 3.8). In contrast, we provide an easy example showing that the rotation of a trajectory of C^∞ vector field around a non-invariant subspace may be

infinite in finite time (Example 3.3).

2. Definition of signed and absolute rotation

2.1. Rotation around an affine subspace

First of all, we define an absolute rotation of the curve γ in $\mathbb{R}^n$ around the origin $0 \in \mathbb{R}^n$. Assuming that γ does not pass through the origin, we can define the spherical image of γ as the curve σ in the unit sphere S^{n-1}, in the following way:

$$\sigma(t) = \frac{\gamma(t)}{\|\gamma(t)\|}.$$

Definition. *The absolute rotation* $R_{abs}(\gamma, 0)$ of the curve γ around the origin is the length of σ, the spherical image of γ in the unit sphere S^{n-1}.

We have the following lemma (easy to check) which relates the rotation to the spherical part of the velocity of the curve:

Lemma 2.1. — *Let $\gamma'(t) = \dfrac{d\gamma(t)}{dt}$ be the velocity vector of $\gamma : I \to \mathbb{R}^n$, and let $\gamma'_r(t)$ and $\gamma'_s(t)$ be the radial and the spherical components of this velocity vector. Then the velocity of the spherical blowing-up σ of γ is:*

$$\sigma'(t) = \frac{\gamma'_s(t)}{\|\gamma(t)\|}.$$

As a consequence, the absolute rotation $R_{abs}(\gamma, 0)$ is given by the integral:

$$R_{abs}(\gamma, 0) = \int_{t \in I} \frac{\|\gamma'_s(t)\|}{\|\gamma(t)\|} \, dt.$$

Remark. The absolute rotation $R_{abs}(\gamma, 0)$ is invariant with respect to the monotone reparametrizations of the curve γ and in the same time we want our results to be about the geometry of the curve and not depending of one of its parametrizations. This is why in what follows we implicitly consider the geometric trajectory $\gamma(I)$ of the injective curve $\gamma : I \to \gamma(I)$ as the class of $\gamma : I \to \mathbb{R}^n$ modulo its injective reparametrizations.

For plane curves γ in $\mathbb{R}^2$ we can define their "signed rotation" $R(\gamma, 0)$ around the origin, which is, of course, the usual rotation index. Indeed, in this case the unit sphere is a circle. Assuming that the orientation of this circle (and of the curve γ) has been chosen, we can define the signed rotation, essentially, by the same expression as above:

Definition. The *signed rotation* $R(\gamma, 0)$ of γ around the origin is defined by:

$$R(\gamma, 0) = \frac{1}{2\pi} \int_{\gamma} \pm \frac{\|\gamma'_s(t)\|}{\|\gamma(t)\|} dt.$$

Here the sign under the integral is chosen according to the direction of the tangent to the unit circle vector $\gamma'_s(t)$.

Remark. Notice the normalization by 2π, the unit circle length, which appears here to make the linking number, defined below, an integer. In codimension greater or equal to three we do not normalize the length of the spherical curves.

An absolute rotation of the curve γ in $\mathbb{R}^n$ around a linear k-dimensional subspace $\mathcal{L} \subset \mathbb{R}^n$ is defined as follows: let $\mathcal{L}^\perp$ denote the orthogonal subspace to $\mathcal{L}$. Let $\tilde{\gamma}$ be the projection of γ on $\mathcal{L}^\perp$. Assuming that γ does not touch $\mathcal{L}$, we get $\tilde{\gamma}$ not passing through the origin in $\mathcal{L}^\perp$.

Definition. The *absolute rotation* $R_{abs}(\gamma, \mathcal{L})$ is defined as the absolute rotation of the curve $\tilde{\gamma}$ in $\mathcal{L}^\perp$ around the origin. In the case of a linear subspace $\mathcal{L}$ of codimension 2 in $\mathbb{R}^n$, a signed rotation $R(\gamma, \mathcal{L})$ of γ around $\mathcal{L}$ is defined as the signed rotation of the curve $\tilde{\gamma}$ in the plane $\mathcal{L}^\perp$ around the origin.

Of course the absolute rotation always bounds from above the absolute value of the signed one.

For a closed curve γ and for a subspace $\mathcal{L}$ of codimension 2 the signed rotation $R(\gamma, \mathcal{L})$ is an integer, and it is a topological invariant. For γ non-closed this signed rotation $R(\gamma, \mathcal{L})$ may take non-integer values. However, it is still an invariant of deformations of γ, preserving the end points, and not touching $\mathcal{L}$.

2.2. Rotation of two curves in $\mathbb{R}^3$

It is well known that the linking number of two closed curves in $\mathbb{R}^3$ can be defined via an integral expression, the so-called Gauss integral (see, for example [Ar-Kh], [Du-Fo-No]). This gives us a natural way to define also an absolute and a signed rotation of two curves (closed or non-closed) one around the other.

For non-closed curves the rotation defined in this (or any other) way cannot be metrically or topologically invariant. But on the other hand, the Gauss integral representation provides a powerful analytic tool for its investigation. Our presentation in the next subsection follows very closely the one given in [Du-Fo-No].

2.2.1. The Linking Coefficient

Consider a pair of smooth, closed, regular directed curves in $\mathbb{R}^3$, which do not intersect. We may assume them to be parametrized in the following way: $\gamma_i : I_i \to \mathbb{R}^3$, with I_1, I_2 two compact intervals. We denote our geometric curves by γ_1 and γ_2 (instead of $\gamma_1(I_1), \gamma_2(I_2)$).

Definition. The *linking coefficient of the two curves* γ_1, γ_2 is defined, in terms of the "Gauss integral", by:

$$\{\gamma_1, \gamma_2\} = \frac{1}{4\pi} \int_{t_1 \in I_1} \int_{t_2 \in I_2} \frac{< \gamma_1'(t_1) \wedge \gamma_2'(t_2), \gamma_{12}(t_1, t_2) >}{\|\gamma_{12}(t_1, t_2)\|^3} \, dt_1 dt_2,$$

where $\gamma_{12}(t_1, t_2) = \gamma_2(t_2) - \gamma_1(t_1)$.

Remark. In this definition the normalization by 4π has to be seen as the normalization by the volume of the unit sphere of $\mathbb{R}^3$ (see the Remark that follows the proof of Theorem 2.2).

Let us stress that this definition immediately shows that $\{\gamma_1, \gamma_2\}$ does not depend on the parametrizaton of the curves nor on their rigid transformations. Intuitively speaking the linking coefficient gives the algebraic (i.e. signed) number of loops of one contour around the other. This interpretation is justified by the following result.

Theorem 2.2. — Let γ_1 and γ_2 be two closed curves in $\mathbb{R}^3$ and assume that $I_1 = [0, 2\pi]$.

(i) *The linking coefficient $\{\gamma_1, \gamma_2\}$ is an integer, and is unchanged by deformations of γ_1 and γ_2, involving no intersection of one curve with the other.*

(ii) *Let $F : D^2 \to \mathbb{R}^3$ be a map of the disc D^2 which agrees with $\gamma_1 : t \mapsto \gamma_1(t),\ 0 \le t \le 2\pi$, on the boundary $\partial D^2 \simeq S^1 \simeq \dfrac{[0, 2\pi]}{0 \sim 2\pi}$, and is transversal to the curve $\gamma_2 \subset \mathbb{R}^3$. Then the "topological linking number" which is the intersection index $F(D^2) \cdot \gamma_2$, (i.e. the number of the intersection points of $F(D^2)$ and γ_2, counted with the signs, reflecting the orientation), is equal to the linking coefficient $\{\gamma_1, \gamma_2\}$.*

Proof. The closed curves $\gamma_i(t),\ i = 1, 2$, give rise to a 2-dimensional, closed, oriented parametric surface $\gamma_1 \times \gamma_2$ in $\mathbb{R}^6$:

$$\gamma_1 \times \gamma_2 : (t_1, t_2) \mapsto (\gamma_1(t_1), \gamma_2(t_2)).$$

Since the curves are non-intersecting the map $\varphi : \gamma_1 \times \gamma_2 \to S^2$, given by:

$$\varphi(t_1, t_2) = \frac{\gamma_1(t_1) - \gamma_2(t_2)}{\|\gamma_1(t_1) - \gamma_2(t_2)\|}$$

is well defined. An easy geometric consideration shows that the integrand in the Gauss integral is just the Jacobian of the map φ. Therefore the Gauss integral above is equal to the degree of the map φ. Hence the linking coefficient is indeed an integer. Under deformations of the curves $\{\gamma_1, \gamma_2\}$ involving no intersection one with the other, the map φ undergoes a homotopy, so that its degree, and therefore also the linking coefficient, are preserved. Let us stress that in the process of these deformations each of the curves $\gamma_1,\ \gamma_2$ separately may cross itself in an arbitrary way. Of course, the topological linking number is also preserved by such deformations.

We now prove (ii). If the curves are not linked (i.e. if by means of a homotopy respecting non-intersection they can be brought to opposite sides of a 2-dimensional plane in $\mathbb{R}^3$) then it can be verified directly that $\{\gamma_1, \gamma_2\} = deg\ \varphi = 0$. In a general case we can "push" γ_1 along γ_2 in such a way that after this deformation it comes close to γ_2 only in a neighborhood of exactly one point. Then by applying another deformation (remind that self-intersections of the curves are allowed) we reduce the general case of the problem of calculating the linking coefficient essentially to the following simple situation: the curve γ_2 is a straight line, while γ_1 is a circle, orthogonal to γ_1 and passed several times in the positive or negative direction. Thus we suppose γ_1 and γ_2 to be given respectively by $\gamma_1(t_1) = (cos\ t_1,\ sin\ t_1, 0),\ 0 \le t_1 \le 2\pi$ and $\gamma_2(t_2) = (0, 0, t_2),\ -\infty < t_2 < \infty$. The linking coefficient for these two curves is:

$$\{\gamma_1, \gamma_2\} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{dt_1 dt_2}{(1 + t_2^2)^{3/2}} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dt_2}{(1 + t_2^2)^{3/2}}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{du}{ch^2(u)} = \frac{1}{2}[th(u)]_{-\infty}^{+\infty} = 1.$$

Hence for these two directed curves the statement (ii) of the Theorem holds. The general result now follows via the deformation described above. □

Remark. One can give another proof of Theorem 2.2. As above, we notice that the Gauss integral is equal to the degree of the mapping $\varphi : \gamma_1 \times \gamma_2 \to S^2$. Now fix a point p in S^2 which is a regular value of φ and consider the projection π along the corresponding line ℓ_p onto the orthogonal plane P_p. The preimages $\varphi^{-1}(p)$ correspond exactly to the crossing points of the plane curves $\pi(\gamma_1)$ and $\pi(\gamma_2)$ in P_p. The sign of the Jacobian of φ at each of the preimages $\varphi^{-1}(p)$ can be computed via the directions of $\pi(\gamma_1)$ and $\pi(\gamma_2)$ at their corresponding crossing point, taking into account, which curve is "above" and which is "below".

Now the degree of the mapping φ is the sum of the signs of the Jacobian of φ over all the preimages $\varphi^{-1}(p)$. On the other hand, the corresponding sum over all the crossing points of $\pi(\gamma_1)$ and $\pi(\gamma_2)$ can be easily interpreted as the topological linking number of γ_1 and γ_2.

2.2.2 Signed and absolute rotation

As the curves γ_1, γ_2 are not necessarily closed, the Gauss integral can be still computed.

Definition. For two curves γ_1, γ_2 in $\mathbb{R}^3$, closed or non-closed, we call the Gauss integral along these curves the *signed rotation of the curves* γ_1 and γ_2 and denote it by $R(\gamma_1, \gamma_2)$. We have:

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi}\int_{t_1 \in I_1}\int_{t_2 \in I_2} \frac{< \gamma_1' \wedge \gamma_2', \gamma_{12} >}{\|\gamma_{12}\|^3} \, dt_1 dt_2,$$

where $\gamma_{12} = \gamma_2 - \gamma_1$.

For two curves γ_1, γ_2, not necessarily closed, the *absolute rotation* of these curves is defined as:

$$R_{abs}(\gamma_1, \gamma_2) = \frac{1}{4\pi}\int_{t_1 \in I_1}\int_{t_2 \in I_2} \frac{| < \gamma_1' \wedge \gamma_2', \gamma_{12} > |}{\|\gamma_{12}\|^3} \, dt_1 dt_2.$$

Remarks. The absolute rotation of two curves bounds the absolute value of their signed rotation:

$$|R(\gamma_1, \gamma_2)| \leq R_{abs}(\gamma_1, \gamma_2).$$

In particular, for γ_1, γ_2 closed the absolute rotation bounds the absolute value of the linking number $\{\gamma_1, \gamma_2\}$.

For γ_1, γ_2 not closed, $R(\gamma_1, \gamma_2)$ does not need to be an integer any more. This rotation number also is not invariant under the deformations of the curves γ_1, γ_2 without crossing one another, even if we assume that the end-points of γ_1 and γ_2 are fixed. Indeed, if we take the curve γ_1 to be a "long" segment of the straight line, and the curve γ_2 to be the unit circle around γ_1, the computation at the end of the proof of Theorem 2.2 above shows that the signed rotation $R(\gamma_1, \gamma_2)$, is approximately one. On the other hand, we can deform the circle γ_2, with one of its points fixed, as follows: we pull it out from the

segment γ_1, and then contract it to the point. The rotation of the deformed curves is zero, so it was not preserved in the process of the deformation.

However, for one of the curves, say γ_1, closed, we have the following result:

Proposition 2.3. — *Let the curve γ_1 be closed. Then the signed rotation $R(\gamma_1, \gamma_2)$ is invariant under the deformations of the curve γ_2 (without crossing γ_1) if the end-points of γ_2 remain fixed.*

Proof. Consider a closed curve $\tilde{\gamma}_2$ obtained from γ_2 by passing it twice in the opposite directions. We have $R(\gamma_1, \tilde{\gamma}_2) = 0$, since the rotation of these two closed curves is invariant under deformation, while $\tilde{\gamma}_2$ can be deformed into the point without crossing γ_1. Now consider another deformation of $\tilde{\gamma}_2$, where one copy of γ_2 remains fixed, while another copy, $\hat{\gamma}_2$, undergoes a deformation without crossing γ_1 and with the end points fixed. The signed rotation remain zero in this deformation, so $R(\gamma_1, \tilde{\gamma}_2) = R(\gamma_1, \gamma_2) - R(\gamma_1, \hat{\gamma}_2) = 0$. Hence $R(\gamma_1, \hat{\gamma}_2)$ remains the same in the deformation. $\square$

Another property which can be obtained by a rather straightforward computation, is the following:

Proposition 2.4. — *For $\gamma_1 = \mathcal{L}$ a straight line, the signed rotation $R(\gamma_1, \gamma_2)$ (resp. the absolute rotation $R_{abs}(\gamma_1, \gamma_2)$) given by the Gauss integral coincides with the signed rotation $R(\gamma_2, \mathcal{L})$ (resp. the absolute rotation $R_{abs}(\gamma_2, \mathcal{L})$) of the curve γ_2 around the straight line $\mathcal{L}$, as defined by projection on $\mathcal{L}^{\perp}$ in Section* 2.1 *above.*

The absolute rotation $R_{abs}(\gamma_1, \gamma_2)$ given by the expression (2.7) coincides with the absolute rotation of the curve γ_2 around the straight line γ_1, as defined in Section 2.1.

Proof. Let us start with the case of the signed rotation. We have, passing to the length parametrization,

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{I_1} \int_{I_2} \frac{< n_1 \wedge n_2, \gamma_{12} >}{\|\gamma_{12}\|^3} \; ds_1 ds_2,$$

where n_1, n_2 are the unit tangent vectors to the curves γ_1, γ_2, and s_1, s_2 are the length parameters on the curves γ_1, γ_2, respectively.

Let $\eta_2(s_2)$ denote the vector joining the point $s_2 \in \gamma_2$ with the projection of s_2 onto the straight line γ_1 (see Figure 1). We have $< n_1 \wedge n_2, \gamma_{12} >=< n_1 \wedge n_2, \eta_2 >$. Hence:

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{I_1} \int_{I_2} \frac{< n_1 \wedge n_2, \eta_2 >}{\|\gamma_{12}\|^3} \; ds_1 ds_2.$$

The vector n_1, being the tangent vector to the straight line γ_1, is constant. Therefore, the triple product under the above integral depends only on s_2 and we have:

$$R(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_{I_2} < n_1 \wedge n_2(s_2), \eta_2(s_2) > \int_{I_1} \frac{ds_1}{\|\gamma_{12}(s_1, s_2)\|^3} \; ds_2.$$

Now the integral:

$$\int_{I_1} \frac{ds_1}{\|\gamma_{12}(s_1, s_2)\|^3}$$

over the straight line γ_1 has been computed (up to a scaling by the distance $\|\eta_2(s_2)\|$ to the line) at the end of Section 2.2.1 above. It is equal to $\dfrac{2}{\|\eta_2(s_2)\|^2}$. So for the rotation $R(\gamma_1, \gamma_2)$ we finally get:

$$R(\gamma_1, \gamma_2) = \frac{1}{2\pi} \int_{I_2} \frac{< n_1 \wedge n_2(s_2), \eta_2(s_2) >}{\|\eta_2(s_2)\|^2} \; ds_2.$$

Now we can replace the vector n_2 in the triple product by its projection $\hat{n}_2$ to the line orthogonal to the lines γ_1 and η_2. Hence this triple product is equal to $\dfrac{\|\hat{n}_2\|}{\|\eta_2\|}$, with the sign defined by the orientation, and we obtain:

$$R(\gamma_1, \gamma_2) = \frac{1}{2\pi} \int_{\gamma_2} \pm \frac{\|\hat{n}_2\|}{\|\eta_2\|} \; ds_2. \tag{1}$$

But η_2 is just the radius-vector of the projection of the curve γ_2 onto the plane orthogonal to the line γ_1, and $\hat{n}_2$ is the orthogonal component of the velocity vector of this projection. Therefore, according to Lemma 2.1 and up to a sign ϵ, the integral (1) is the signed rotation $R(\gamma_2, \mathcal{L})$ of γ_2 around the straight line $\mathcal{L} = \gamma_1$, as defined in Section 2.1 (the sign ϵ is + when in the computation of $R(\gamma_2, \mathcal{L})$ we have oriented the plane $\mathcal{L}^\perp$ in such a way that if $(\vec{u}, \vec{v})$ is an oriented orthonormal basis of $\mathcal{L}^\perp$, $\vec{u} \wedge \vec{v} = -n_1$ (see Figure 1)). This completes the proof of the Proposition for the signed rotation.

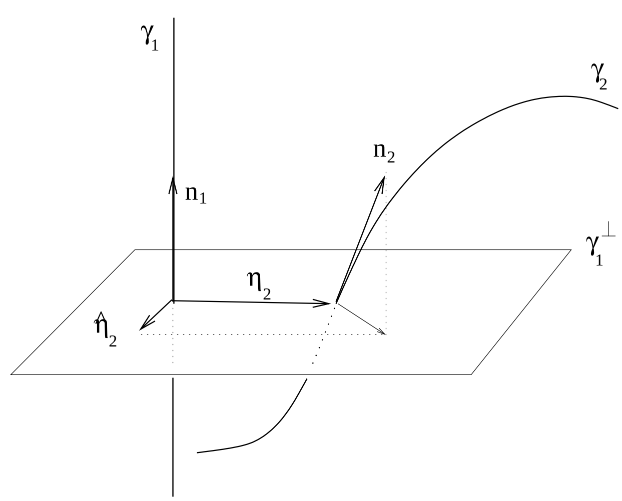

Figure 1

The proof for the absolute rotation is exactly the same: we just take the absolute value of the triple product in each step. Since the proof above consisted of a chain of point-wise equalities of the integrands, it remains valid also for the absolute values. $\square$

3. Rotation of trajectories of Lipschitz vector fields

For a Lipschitz vector field in $\mathbb{R}^n$, the very simple and basic fact is that the angular velocity of its trajectories with respect to any *stationary* point is bounded by the Lipschitz constant K. As an immediate consequence, the rotation of a trajectory around a stationary point is bounded (up to some universal constant C) by the Lipschitz constant K and the time interval T. The bound has this form: $C \cdot K \cdot T$.

Below we remind the proof of this fact. In fact, we show that the same is true for the rotation speed of the trajectories around any linear subspace $\mathcal{L} \subset \mathbb{R}^n$, which is an invariant submanifold of our field.

The main result of this section (Theorem 3.4) is that the "rotation" of a trajectory around any point (*stationary or not*) is bounded in terms of K and T. As the consequence we prove (Theorem 3.8) that the mutual rotation of any two trajectories of a Lipschitz vector field is essentially bounded by the Lipschitz constant and the length of the time interval. More accurately, the absolute rotation of any two trajectories of a Lipschitz vector field on the time intervals T_1 and T_2, respectively, is bounded from above by a linear combination of the expressions $K \cdot \min(T_1, T_2)$ and $C \cdot K^2 \cdot T_1 \cdot T_2$. Easy examples, given in the end, show that this bound is sharp.

At this point, let us remind that the rotation of a curve $\omega : I \to \mathbb{R}^n$ around a point, an affine space or another curve, as defined above, only depends on the trajectory $\omega(I)$, provided we only admit injective (parametrization of) curves. Consequently our bounds not really depend on the field, but rather on the geometry of the trajectories of the fields. This is obvious when we look at the type of bounds we obtain: (up to some universal constant) they are combinations of $K \cdot T$ and $K^2 \cdot T_1 \cdot T_2$, expressions that are unchanged with respect to any transformation of the field that preserve the trajectories.

For Lipschiz vector fields, we can summarize the situation by the following slogan: *"two trajectories have finite mutual rotation in finite time"*.

We cannot expect bounds of this sort to be true for a rotation around a non-invariant subspace of the field. Indeed, the easy-to-construct Example 3.3 below shows that a trajectory of a C^∞-vector field in $\mathbb{R}^3$ can make in *finite* time an *infinite* number of turns around a straight line.

Let us remind that it was shown (in some special cases) in [Gr-Yo] that for a trajectory of a *polynomial* vector field (trajectory which is not in general in some o-minimal structure), its rotation rate around any algebraic submanifold is bounded in terms of the degree of the submanifold and the degree and size of the vector field. As a consequence, we obtain also a linear in time bound on the number of intersections of the trajectory with any algebraic hypersurface.

On the other hand, our bounds on the rotation rate of the trajectories of a polynomial vector field imply upper bounds on the multiplicities of the local intersections of such trajectories with algebraic submanifolds in terms of the degree only.

As the Example 3.3 shows, nothing of this sort can be expected even for C^∞ (and of course, for Lipschitz) vector fields. Still, the results of this section show that Lipschitz

vector fields exhibit rather strong non-oscillation patterns. As far as the rotation of the trajectories of such fields around non-invariant submanifolds is concerned, our current understanding is far from being sufficient. In particular, there is a serious gap between the result of Theorem 3.4 below that a "global" rotation rate of a trajectory of a Lipschitz vector field around a *non-stationary point* is still bounded, and the Example 3.3, demonstrating an infinite rotation around a non-invariant straight line.

3.1. Rotation of a trajectory around a stationary point

Let v be a vector field defined in a certain domain U in $\mathbb{R}^n$. We shall always assume v to satisfy a Lipschitz condition with the constant K:

$$\|v(x) - v(y)\| \leq K\|x - y\|, \quad \text{for any two points} \quad x, y \in U$$

Let $x_0 \in U$ be a stationary or a singular point of v, ie $v(x_0) = 0$, so that the constant curve $c(t) = x_0, t \in \mathbb{R}$, is the integral curve of v passing through x_0. Then for any $x \in U,\ x \neq x_0$, the angular velocity of the trajectory of v, passing through x, with respect to x_0, is equal to $\|\hat{v}(x)\|/\|x - x_0\|$, where $\hat{v}(x)$ is the projection of the vector $v(x)$ to the hyperplane orthogonal to $x - x_0$. Hence, this angular velocity does not exceed K:

$$\frac{\|\hat{v}(x)\|}{\|x - x_0\|} \leq \frac{\|v(x)\|}{\|x - x_0\|} = \frac{\|v(x) - v(x_0)\|}{\|x - x_0\|} \leq K$$

where K is the Lipschitz constant of v. By Lemma 2.1 we obtain :

Proposition 3.1. — *For any trajectory $\omega(t)$ of the field v, its rotation around the stationary point x_0 of the field between the time moments t_1 and t_2, i.e., the length of the spherical curve $s(t) = \dfrac{\omega(t) - x_0}{\|\omega(t) - x_0\|}$ between t_1 and t_2, does not exceed $K \cdot (t_2 - t_1)$.*

Remark. Of course on a non-finite time interval, a trajectory may have infinite local rotation around a stationary point, as shown by the example given below (we avoid the obvious example of a cyclic trajectory, since we aim to work with injective trajectories).

Let us consider the following algebraic field in $\mathbb{R}^2$ with singular point $O = (0, 0)$:

$$v(x, y) = \Big((x^2 + y^2 - 1)x - y, (x^2 + y^2 - 1)y + x\Big),$$

and introduce the following notations: $r^2 = x^2 + y^2$, $\vec{\tau} = (x, y)/r$ and $\vec{n} = (-y, x)/r$. We have :

$$v(x, y) = r(r^2 - 1) \cdot \vec{\tau} + r \cdot \vec{n}.$$

A trajectory passing at a point p with $r(p) = 1$ has to be the unit circle. Now consider $\omega = (\alpha, \beta)$ an integral curve of v passing through a point q with $r(q) < 1$. This trajectory cannot go outside the open unit disc and as we have: $\dfrac{d[r^2 \circ \omega]}{dt} = 2(\alpha\alpha' + \beta\beta')$, we obtain: $\dfrac{d[r^2 \circ \omega]}{dt} = 2(r^2 - 1)(\alpha^2 + \beta^2) < 0$. This proves that the trajectory ω has the singular point O as limit (while the unit circle has to be a limit cycle of this trajectory). Furthermore we know that from the point q, the limit O is approached on an infinite time interval I.

On the other hand, the velocity $\widetilde{v}$ of the spherical blowing-up σ of ω is: $\dfrac{1}{r} \cdot r \cdot \vec{n} = \vec{n}$. We conclude that :

$$R_{abs}(\omega, O) = \int_I \|\vec{n}(w(t))\| \; dt = \int_I 1 \; dt = +\infty.$$

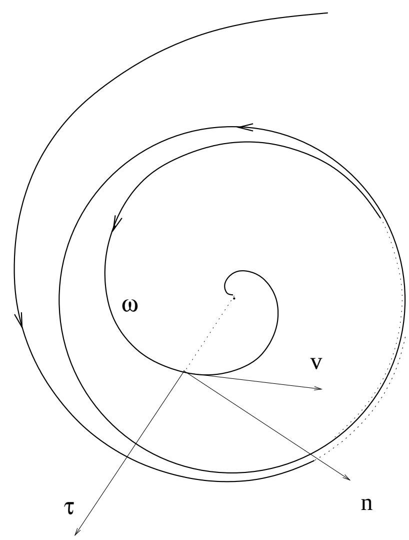

Figure 2

To finish with this remark, let us recall that such an example of spiraling trajectory does not exist for v a gradient vector field of an analytic map, as proved in [Ku-Mo-Pa], [Ku-Mo]; for such a field the length of σ is finite and the limit of the secants passing through the singular point does exist (Thom's Gradient Conjecture).

Exactly in the same way as in Proposition 3.1 we prove the following more general result about rotation of a curve around an invariant affine space $\mathcal{L}$. The proof actually shows that in Proposition 3.1 the assumption $v(x_0) = 0$ has preferably to be considered as "*$v(x_0)$ is tangent to the submanifold $\{x_0\}$ of $\mathbb{R}^n$*", or "*v is a stratified field with respect to the stratification $(\{x_0\}, U \setminus \{x_0\})$ of U*".

Proposition 3.2. — *For any affine subspace $\mathcal{L} \subset \mathbb{R}^n$ which is invariant for the vector field v (i.e. for any $x \in \mathcal{L}$, $v(x)$ is tangent to $\mathcal{L}$ or, in other words, v is stratified with respect to $(\mathcal{L}, U \setminus \mathcal{L})$), the rotation speed of v in the orthogonal to $\mathcal{L}$ direction is bounded by K. In particular, the absolute rotation of any trajectory ω of the field v around $\mathcal{L}$ in time T does not exceed $K \cdot T$.*

Proof. According to the definition of the absolute rotation around a linear subspace (Section 2.1), we consider the orthogonal to $\mathcal{L}$ component $\tilde{v}(x)$ of the vector field $v(x)$. Let x_0 be the projection of x onto $\mathcal{L}$. We denote by $\hat{v}(x)$ the "rotation" component of $\tilde{v}(x)$, orthogonal to $x - x_0$. Then the rotation speed of v in the orthogonal to $\mathcal{L}$ direction is equal to $\|\hat{v}(x)\| / \|x - x_0\|$. Hence, this rotation speed is bounded as follows:

$$\frac{\|\hat{v}(x)\|}{\|x - x_0\|} \leq \frac{\|\tilde{v}(x)\|}{\|x - x_0\|} = \frac{\|\tilde{v}(x) - \tilde{v}(x_0)\|}{\|x - x_0\|} \leq \frac{\|v(x) - v(x_0)\|}{\|x - x_0\|} \leq K.$$

Here we use the fact that $\mathcal{L}$ is invariant for v and hence $\tilde{v}(x_0) = 0$. By Definition we obtain that the absolute rotation of any trajectory $w(t)$ of the field v around $\mathcal{L}$ in time T does not exceed $K \cdot T$. This completes the proof of Proposition 3.2. $\square$

If x_0 is not a stationary point of v, the angular velocity of $v(x)$ with respect to x_0 tends to infinity as x approaches x_0 in any direction transverse to $v(x_0)$. Indeed, as x approaches x_0, $v(x)$ tends to $v(x_0) \neq 0$. By Lemma 2.1, the angular velocity of $v(x)$ with respect to x_0 is equal to $\|\hat{v}(x)\|/\|x - x_0\|$, where $\hat{v}(x)$ is the component of $v(x)$ orthogonal to $x - x_0$. So if x approaches x_0 in a direction transverse to $v(x_0)$, $\hat{v}(x)$ tends to $\hat{v}_0 \neq 0$, while $x - x_0$ tends to zero. See Figure 3.

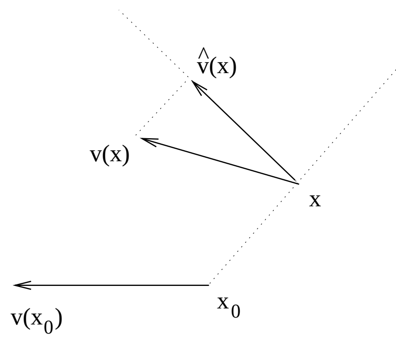

Figure 3

However, one can show (see Theorem 3.4 below) that for any trajectory $\omega(t)$ of v, and for $t_2 - t_1$ big enough, the length of the spherical curve $s(t) = \omega(t) - x_0/\|\omega(t) - x_0\|$ (i.e. the absolute rotation of $\omega(t)$ around x_0) is still bounded by $C \cdot K \cdot (t_2 - t_1)$. The intuitive explanation is as follows: as the trajectory $\omega(t)$ passes very close to x_0, its rotation around x_0 in the time interval $[t_1, t_2]$ comes to approximately 1/2 (see Figure 4).

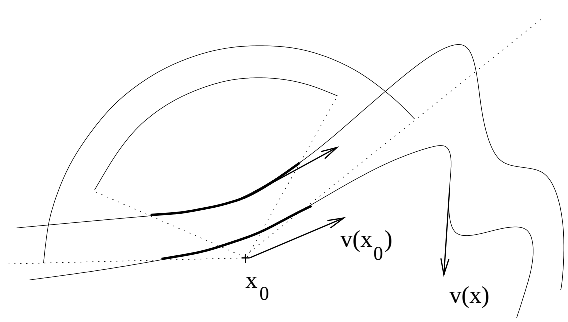

Figure 4

However, continuing to move in the same direction, the trajectory $\omega(t)$ cannot gain more rotation. So for the total rotation of the trajectory $\omega(t)$ around x_0 to grow, its velocity vector $v(\omega(t))$ must change its direction. On the other side, if we know a priori that *the directions of $v(x)$ and $v(x_0)$ are strongly different with respect to $K \cdot \|x - x_0\|$*,

then the rotation speed satisfies the same Lipschitz upper bound as above. Consequently, the time interval $[t_1, t_2]$ has to be large in order to gain a large total rotation.

Although the rotation in finite time of a trajectory of a Lipschitz field around a point is bounded, the rotation of a trajectory of a Lipschitz vector field v around a straight line, *which is not invariant under* v, can be unbounded during a finite interval of time, and this phenomena may occur even for v a C^∞-vector field. Consider for instance the following field:

Example 3.3. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism, defined by:

$$\Phi(x_1, x_2, x_3) = (x_1, x_2, x_3), \quad x_1 \leq 0,$$

$$\Phi(x_1, x_2, x_3) = (x_1, x_2 + \omega_1(x_1), x_3 + \omega_2(x_1)), \quad x_1 \geq 0,$$

where $\omega_1(x_1) = e^{-1/x_1^2} \cos(1/x_1)$, and $\omega_2(x_1) = e^{-1/x_1^2} \sin(1/x_1)$.

One can easily check that Φ is a C^∞-diffeomorphism of a neighborhood of $0 \in \mathbb{R}^3$. Now the image of the positive x_1-semiaxis under Φ is a line w, which makes an infinite number of turns around Ox_1 in any neighborhood of the origin.

Consider the vector field v in $\mathbb{R}^3$, which is an image under Φ of the constant vector field $e_1 = (1, 0, 0)$. Clearly, w is a trajectory of the C^∞-vector field v, and it makes an infinite number of turns around Ox_1 in finite time. In coordinates,

$$v(x_1, x_2, x_3) = D\Phi_{(x_1,x_2,x_3)}(e_1) = \frac{\partial \Phi}{\partial x_1}(x_1, x_2, x_3) = (1, \omega_1'(x_1), \omega_2'(x_1)).$$

Notice that in this example, the orthogonal components of v on the line Ox_1 itself has an infinite number of sign changes, accumulating to the origin.

Remark. Notice that the rotation of any two trajectories of the vector field of the Example 3.3 one around another is zero. Indeed, the field v does not depend on the coordinates x_2, x_3. So the vector joining the intersection points of the two trajectories with the planes $x_1 = c$ remains constant. In particular, this shows that we cannot expect any "transitivity" in the rotation of three curves: take two trajectories of v "far away" from the line Ox_1, while the third trajectory is w as in Example 3.3.

3.2. Rotation of a trajectory around a non-stationary point

As it was mentioned above, one cannot bound uniformly the *momentary angular velocity* of trajectories of a vector field v with respect to a *non-stationary* point x_0. We shall show in this section that nevertheless the "long-time" rotation rate of trajectories of a Lipschitz field v *with respect to any point* x_0, stationary or non-stationary, is uniformly bounded.

Our main result is the following Theorem:

Theorem 3.4. — *Let v be a Lipschitz vector field on an open set $U \subset \mathbb{R}^n$ with Lipschitz constant K. Let $\omega(t)$ be a trajectory of v. Then for any $x_0 \in U$ the absolute rotation of ω around x_0 between any two time moments t_1 and t_2 satisfies:*

$$R_{abs}(\omega, x_0) \leq 4 + K \cdot (t_2 - t_1).$$

Proof. To prove Theorem 3.4 we need a general geometric lemma, which expresses accurately (in one of many possible ways) the intuitively clear fact that a long curve inside a fixed sphere must oscillate. Let $\omega(t)$ be a $\mathcal{C}^1$-curve (or piecewise $\mathcal{C}^1$) in the unit sphere $S^{n-1} \subset \mathbb{R}^n$. For any two time moments t_1 and t_2 denote by $s(t_1, t_2)$ the length of the curve ω between t_1 and t_2.

For each "equator circle" λ in S^{n-1} consider the "longitude" projection $\pi : S^{n-1} \to \lambda$ of the sphere (without the poles) on the circle λ. A circle λ is given as the intersection of S^{n-1} with a 2-dimensional vector plane Λ of $\mathbb{R}^n$. The fiber of π over a point $t \in \lambda$ lies in a $(n-2)$-dimensional sphere $S^{n-2}_{\lambda,t} \subset S^{n-1}$ which is normal to λ and characterized as the intersection $H \cap S^{n-1}$, where H is the hyperplane of $\mathbb{R}^n$ passing through t and containing $\Lambda^{\perp}$. The projection of a point $x \in S^{n-1}$ is defined as soon as x is not in the poles, that is to say $x \notin \Lambda^{\perp} \cap S^{n-1} \simeq S^{n-3}$(see Figure 5). Since ω is (piecewise) $\mathcal{C}^1$, by Sard's theorem, it does not intersect a generic sub-sphere $S^{n-3} \subset S^{n-1}$. Hence, the projection $\pi : \omega \to \lambda$ is defined and regular for generic circles λ.

For the computations below it is convenient to fix a real parameter $\theta > 4$.

Lemma 3.5. — *Let $R > 0$ and S^{n-1}_R be the sphere of $\mathbb{R}^n$ of radius R. If the length $s(t_1, t_2)$ of a curve $\omega \subset S^{n-1}_R$ between t_1 and t_2 satisfies $s(t_1, t_2) > 2\pi R \cdot \theta$ then there exist an "equator circle" λ in S^{n-1} and time moments τ_1 and τ_2, with $t_1 < \tau_1 < \tau_2 < t_2$, such that the following is true:*

(i) *The "longitude" projections $\pi(\omega(\tau_1))$ and $\pi(\omega(\tau_2))$ of the points $\omega(\tau_1)$ and $\omega(\tau_2)$ on the circle λ coincide or are antipodal points on λ.*

(ii) *Denoting by ℓ the tangent line to λ at this common projection point or antipodal points, the norm of the projections of the velocity vectors $v(\tau_i) = \dfrac{d\omega}{dt}(\tau_i),\ i = 1, 2$ on the line ℓ are $\geq \dfrac{\theta - 4}{4\theta} \cdot \dfrac{s(t_1, t_2)}{t_2 - t_1}$, and the directions of these projections are opposite in ℓ (in case $\pi(\omega(\tau_1))$ and $\pi(\omega(\tau_2))$ are not equal but antipodal, $v(\tau_2)$ is identified with its projection on the tangent line at $\pi(\omega(\tau_1))$).*

We prove Lemma 3.5 below in this section. Let us complete now the proof of Theorem 3.4.

We apply Lemma 3.5 to the spherical curve $\sigma(t) = (\omega(t) - x_0)/\|\omega(t) - x_0\|$, which is contained in the unit sphere S^{n-1} in $\mathbb{R}^n$. By this lemma either the length $s(t_1, t_2)$ of $\sigma(t)$ between the moments t_1 and t_2 is smaller than $2\pi \cdot \theta$, or there is an equator circle $\lambda \subset S^{n-1}$ and time moments τ_1, τ_2 such that the longitude projections $\pi(\sigma_1)$ and $\pi(\sigma_2)$ of the points $\sigma_1 = \sigma(\tau_1), \sigma_2 = \sigma(\tau_2)$ coincide or are antipodal points on λ. For instance, let us assume that $\pi(\sigma_1) = \pi(\sigma_2) = \mu$, the proof being the same in the case $\pi(\sigma_1)$ and $\pi(\sigma_2)$ are antipodal in λ, because the tangent lines to λ at antipodal points are the same. By Lemma 3.5, the orthogonal projections of the velocity vectors of σ at $\tau_i,\ i = 1, 2$ on ℓ, the tangent line to λ at μ, are in norm $\geq \dfrac{\theta - 4}{4\theta} \cdot \dfrac{s(t_1, t_2)}{t_2 - t_1}$ and the directions of these two projections are opposite.

Denoting the corresponding points $\omega(\tau_1)$ and $\omega(\tau_2)$ of the original trajectory $\omega(t)$ by

w_1 and w_2, we have: $\sigma_1 = \dfrac{w_1 - x_0}{\|w_1 - x_0\|}$ and $\sigma_2 = \dfrac{w_2 - x_0}{\|w_2 - x_0\|}$. Let $v(t)$ denote, as above, the velocity vector of the trajectory $\omega(t)$, while $\tilde{v}(t)$ denotes the velocity vector of the spherical curve σ. The vector $\tilde{v}(t)$ is the orthogonal to $\omega(t) - x_0$ component of $\dfrac{v(t)}{\|\omega(t) - x_0\|}$.

A simple property of the longitude projection is the following (see Figure 5): if for a point $s \in S^{n-1}$ we denote by $T(s)$ the tangent hyperplane to S^{n-1} at s (which is also the orthogonal hyperplane to the radius -vector of s), and by $\ell(\pi(s))$ the tangent line to the circle λ at the longitude projection $\pi(s)$ of s to λ, then $\ell(\pi(s)) \subset T(s)$. In particular, denoting by T_1, T_2 the tangent hyperplanes to S^{n-1} at σ_1 and σ_2, respectively, we have $\ell \subset T_1,\ \ell \subset T_2$.

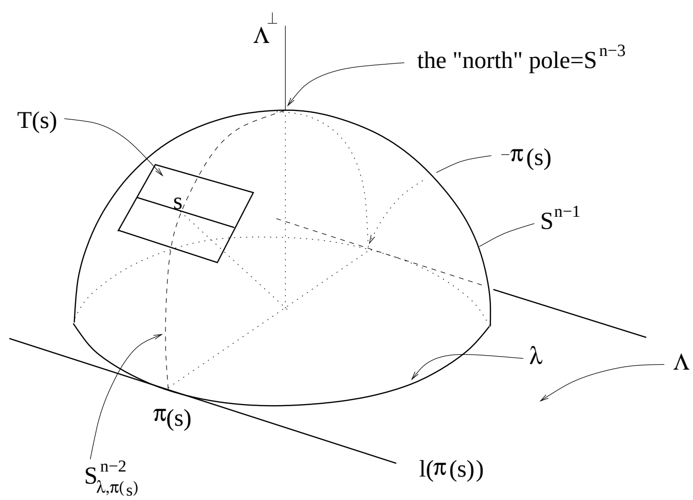

Figure 5

The immediate consequence of this inclusion is that the orthogonal projections on ℓ of the velocity vectors $\tilde{v}_1$ and $\tilde{v}_2$ of the spherical curve σ are the same as the projections on ℓ of the normalized velocity vectors $\dfrac{v_1}{\|w_1 - x_0\|}$ and $\dfrac{v_2}{\|w_2 - x_0\|}$ of the original trajectory ω. Consequently, these two projections are $\geq \dfrac{\theta - 4}{4\theta} \cdot \dfrac{s(t_1, t_2)}{t_2 - t_1}$ and their directions are opposite.

Let us now consider two auxiliary vectors: $V_1 = \dfrac{v_1 - v(x_0)}{\|w_1 - x_0\|}$ and $V_2 = \dfrac{v_2 - v(x_0)}{\|w_2 - x_0\|}$. Since the projections of $\dfrac{v_1}{\|w_1 - x_0\|}$ and $\dfrac{v_2}{\|w_2 - x_0\|}$ on ℓ had opposite directions, independently of the vector $v(x_0)$ at least one of the vectors V_1, V_2 has its projection on ℓ of size $\geq \dfrac{\theta - 4}{4\theta} \cdot \dfrac{s(t_1, t_2)}{t_2 - t_1}$. But since the vector field v is Lipschitz, the norms of both the vectors V_1 and V_2 are bounded by K. Hence $\dfrac{\theta - 4}{4\theta} \cdot \dfrac{s(t_2, t_1)}{t_2 - t_1} \leq K$, or $s(t_2, t_1) \leq \dfrac{4\theta}{\theta - 4} \cdot K \cdot (t_2 - t_1)$. In any case we have proved :

$$R_{abs}(\omega, x_0) = s(t_2, t_1) \leq \max(\theta, \frac{4\theta}{(\theta - 4)} \cdot K \cdot (t_2 - t_1)).$$

But this bound is true for any $\theta > 4$, and for $\theta = 4 + K \cdot (t_2 - t_1)$, we have both expressions under the maximum sign equal one another. Therefore, for this specific choice of $\theta > 4$ we get:

$$\max(\theta, \frac{4\theta}{(\theta - 4)} \cdot K \cdot (t_2 - t_1)) = 4 + K \cdot (t_2 - t_1)$$

Up to the proof of Lemma 3.5, this completes the proof of Theorem 3.4. $\square$

Remark. We could complete the proof of Theorem 3.4 without returning to the point x_0, just by comparing the vectors v_1 and v_2 and using the Lipschitz property of the vector field v. However, the comparison with the vector $v(x_0)$ used in the proof above, can be applied also in more general situations, like estimating rotation of a Lipschitz vector field around a non-invariant subspace.

Proof of Lemma 3.5. Let $\omega(t)$ be a $\mathcal{C}^1$-curve (possibly piecewise $\mathcal{C}^1$) in S_R^{n-1} and assume that ω is defined on the time interval $[t_1, t_2]$. We denote, as above, by $s(t_1, t_2)$ the length of ω (between t_1 and t_2). For each one dimensional circle $\lambda \simeq S_R^1$ in S_R^{n-1}, consider the longitude projection $\pi_\lambda : S_R^{n-1} \to \lambda$. For $t \in \lambda$ we denote by $S_{\lambda,t}^{n-2}$ the $(n-2)$-sphere of S_R^{n-1} which contains the fiber $\pi_\lambda^{-1}(t)$. We denote by $\mathcal{O}(n)$ the orthogonal group of $\mathbb{R}^n$ and by θ_n its Haar measure. For $S \simeq S_R^1$ a fixed circle in S_R^{n-1}, $g \in \mathcal{O}(n)$ and $t \in S$, we denote by $S_{g\cdot S, g\cdot t}^{n-2}$ the $(n-2)$-sphere of S_R^{n-1}, which contains the fiber over $g \cdot t$ of the longitude projection onto $g \cdot S$. By the spherical Cauchy-Crofton formula (see [Fe 2] 3.2.48), we have:

$$\frac{1}{\pi R} \cdot s(t_1, t_2) = \int_{g \in \mathcal{O}(n)} \#(\omega \cap S_{g\cdot S, g\cdot t}^{n-2})\ d\theta_n(g)$$

Now by Fubini's theorem:

$$2 \cdot s(t_1, t_2) = \int_{t \in S} \int_{g \in \mathcal{O}(n)} \#(\omega \cap S_{g\cdot S, g\cdot t}^{n-2})\ d\theta_n(g)\ dt$$

$$= \int_{g \in \mathcal{O}(n)} \int_{t \in S} \#(\omega \cap S_{g\cdot S, g\cdot t}^{n-2})\ dt\ d\theta_n(g)$$

We conclude that for some $g \in \mathcal{O}(n)$,

$$\int_{t \in S} \#(\omega \cap S_{g\cdot S, g\cdot t}^{n-2})\ dt \geq 2 \cdot s(t_1, t_2)$$

But this integral is exactly twice the length of the projection of ω onto the circle $\lambda = g \cdot S$.

So we have shown that for any curve $\omega \subset \mathbb{R}^n$ there is a circle $\lambda \in S_R^{n-1}$ such that the projection of ω onto λ has its length $\geq s(t_1, t_2)$. But by our assumption, we also have:

$$s(t_1, t_2) > 2\pi R \cdot \theta$$

Consequently, to prove Lemma 3.5 it is enough to prove the following Lemma 3.6, which is Lemma 3.5 in the case of a curve ω contained in a circle S_R^1:

Lemma 3.6. — *Let θ be any real number > 4 and ω be a $\mathcal{C}^1$ (possibly, piecewise $\mathcal{C}^1$) curve in S^1_R, given with an orientation. Let us assume that its length $s(t_1, t_2)$ between the time moments t_1 and t_2 is $> 2\pi R \cdot \theta$. Then there exist time moments τ_1 and τ_2, $t_1 < \tau_1 < \tau_2 < t_2$, such that the following is true:*

(i) *The points $\omega(\tau_1)$ and $\omega(\tau_2)$ coincide or are antipodal points on S^1_R.*

(ii) *The velocity vectors $v(\tau_i) = \dfrac{d\omega}{dt}(\tau_i)$, $i = 1, 2$ satisfies:*

- $\|v(\tau_i)\| \geq \dfrac{1}{4}\dfrac{s(t_1, t_2)}{t_2 - t_1}$ *in the case of a closed curve,*
- $\|v(\tau_i)\| \geq \dfrac{\theta - 4}{4\theta} \cdot \dfrac{s(t_1, t_2)}{t_2 - t_1}$ *in the case of a non-closed curve,*
- *the directions of $v(\tau_1)$ and $v(\tau_2)$ are opposite on the tangent line of S^1_R at $\omega(\tau_1)$ and $\omega(\tau_2)$ (in case $\omega(\tau_1)$ and $\omega(\tau_2)$ are not equal but antipodal, $v(\tau_2)$ is identified with its projection on the tangent line at $\omega(\tau_1)$).*

Remark. Since for any $\theta > 4$ we always have $\dfrac{\theta - 4}{4\theta} \leq \dfrac{1}{4}$, in Lemma 3.6.(ii) we have in both case (the curve being closed or not): $\|v(\tau_i)\| \geq \dfrac{\theta - 4}{4\theta} \cdot \dfrac{s(t_1, t_2)}{t_2 - t_1}$, as required by Lemma 3.5.

Proof of Lemma 3.6. The curve ω can be considered as "lying over" the circle S^1_R. Let $s \in S^1_R$, $t \in \omega^{-1}(s)$ and let us denote $\alpha = \dfrac{4}{\theta - 4}$ and $T = t_2 - t_1$ the time interval.

First of all, we can always reduce the situation to the case of a closed curve ω. Indeed, if ω is not closed, but is contained in S^1_R, we just "close up" this curve with a circular curve joining the endpoints, passed with the velocity $\dfrac{2\pi R}{\alpha T}$. We denote the new length of our curve after the "closing up" process by $\widetilde{L}$. We observe that the new time interval $\widetilde{T}$ satisfies: $\widetilde{T} \leq (1 + \alpha)T$. Assuming that Lemma 3.6 is true *for closed curves*, for the new closed curve we therefore find some points with the velocity at least:

$$\frac{\tilde{L}}{4\widetilde{T}} \geq \frac{s(t_1, t_2)}{4\widetilde{T}} \geq \frac{s(t_1, t_2)}{4(1 + \alpha)T} = \frac{\theta - 4}{4\theta} \cdot \frac{s(t_1, t_2)}{t_2 - t_1}.$$

On the other hand, by assumptions we have

$$\frac{s(t_1, t_2)}{4(1 + \alpha)T} > \frac{2\pi R \cdot \theta}{4(1 + \alpha)T} = \frac{2\pi R}{\alpha T}.$$

Hence, by the above choice of the velocity on the added interval, the points found have to belong to the original curve, and not to the added interval.

So from now on we assume that $\omega(t_1) = \omega(t_2)$ and since we are allowed to deal with antipodal points and compare the velocity vectors at these points, we can assume in what follows that ω is homotopically trivial in S^1_R. So it can be considered as "lying over $\mathbb{R}$". Remind that the total length of ω over a subset $A \subset \mathbb{R}$ is equal to the integral $\int_A N(s)ds$ of the number $N(s)$ of the points of ω over A. The same is true for the "positive" and the "negative length" of ω, since ω is homotopically trivial in S^1_R and thus may be view as a closed curve in a segment. On the other hand, since $\omega(t_1) = \omega(t_2)$, ω covers exactly

the same length both in the positive and in the negative directions. The same is true over any subset $A \subset \mathbb{R}$. Therefore, over each subset $A \subset \mathbb{R}$ the "positive length" of ω is equal to its "negative length".

Now let us assume that the statement of Lemma 3.6 is not true for the curve ω. Denote its length $s(t_1, t_2)$ by L. We thus assume there is no point in w over which the velocities in the opposite directions are both $\geq L/4T$. Consequently, over each point $s \in S^1_R$ either all the positive velocities of ω are $< L/4T$, or all the negative velocities of ω are $< L/4T$, or both.

Denote by A_1 (respectively, A_2, A_3) the sets of points in S^1_R where the first (respectively, the second or the third) alternative holds. Let us take those of the sets A_1, A_2 over which the total length of ω is larger, say, A_1, and let $A = A_1 \cup A_3$. The total length of ω over A is $\geq L/2$.

By the remark above, the "positive length" of ω over A is equal to its "negative length", and hence each is $\geq L/4$. But by the construction, at each positive point of ω over A the velocity is $< L/4T$. Therefore, the total time required for ω to cover A in the positive direction is $> \dfrac{L/4}{L/4T} = T$. This contradiction proves Lemma 3.6, thus it also proves Lemma 3.5 and finishes the proof of Theorem 3.4. $\square$

The Euclidean version of Lemma 3.5 is obtained by using the Euclidean integral-geometric Cauchy-Crofton formula ([Fe 1] 5.11, [Fe 2] 2.10.15).

Let us define the constant C_n by $C_n = c_n V_n$, where c_n is the constant in the Cauchy-Crofton formula for curves in $\mathbb{R}^n$, and V_n is the volume of the unit sphere in $\mathbb{R}^n$. We have explicitly $c_n = \Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{n}{2})$, and $V_n = 2\Gamma^n(\frac{1}{2})/\Gamma(\frac{n}{2})$, where Γ is the Euler function. With these notations, let us state the Euclidean version of Lemma 3.5.

Proposition 3.7. — *Let ω be a (piecewise) $\mathcal{C}^1$ curve in a ball of radius R of $\mathbb{R}^n$ and let, $\theta > 8$ be a real number. If the length $s(t_1, t_2)$ of the curve ω between t_1 and t_2 satisfies $s(t_1, t_2) > \theta \cdot C_n \cdot R$, then there exist a straight line λ and time moments τ_1 and τ_2, $t_1 < \tau_1 < \tau_2 < t_2$, such that the following is true:*

(i) *The orthogonal projections $\pi(\omega(\tau_1))$ and $\pi(\omega(\tau_2))$ of the points $\omega(\tau_1)$ and $\omega(\tau_2)$ on the line λ coincide.*

(ii) *The norms of the orthogonal projections of the velocity vectors $v(\tau_i) = \dfrac{d\omega}{dt}(\tau_i)$, $i = 1, 2$ on the line λ are $> \dfrac{\theta - 8}{4\theta} \cdot \dfrac{s(t_1, t_2)}{t_2 - t_1}$, and the directions of these projections are opposite.*

3.3. Rotation of two Lipschitz trajectories

In this section we prove that the absolute rotation of any two trajectories of a Lipschitz vector field in $\mathbb{R}^3$ is bounded in terms of the Lipschitz constant K and the time interval.

Theorem 3.8. — *Let ω_1 and ω_2 be trajectories, on time intervals T_1 and T_2 respectively, of a Lipschitz vector field v defined in some open subset U of $\mathbb{R}^3$. Then the*

mutual absolute rotation of w_1 and w_2 satisfies:

$$R_{abs}(\omega_1, \omega_2) \leq \frac{K}{\pi} \cdot \min(T_1, T_2) + \frac{1}{4\pi} \cdot K^2 \cdot T_1 \cdot T_2,$$

where K is the Lipschitz constant of v.

In fact, we shall prove a more accurate version of this theorem, which bounds the absolute rotation $R_{abs}(\omega_1, \omega_2)$ through the Lipschitz constant of the field and through the bound of the absolute rotation of the trajectory ω_1 (respectively ω_2) around the *points* of ω_2 (respectively of ω_1) (Theorem 3.9). Then to get back to Theorem 3.8 we use the uniform bound on the rotation of Lipschitz trajectories around points, provided by Theorem 3.4.

Theorem 3.9. — *Let ω_1 and ω_2 be trajectories, on time intervals T_1 and T_2 respectively, of a Lipschitz vector field v defined in some open subset U of $\mathbb{R}^3$. With the following notations:*

$$R_1 = \max_{p_2 \in w_2} R_{abs}(\omega_1, p_2), \quad R_2 = \max_{p_1 \in w_1} R_{abs}(\omega_2, p_1),$$

the absolute rotation of w_1 and w_2 satisfies:

$$R_{abs}(\omega_1, \omega_2) \leq \frac{1}{4\pi} \cdot K \cdot R_1 \cdot T_2 \ \ \text{and} \ \ R_{abs}(\omega_1, \omega_2) \leq \frac{1}{4\pi} \cdot K \cdot R_2 \cdot T_1.$$

Corollary 3.10. — *With the same notations and hypothesis as in Theorem 3.9, we have:*

$$R_{abs}(\omega_1, \omega_2) \leq \frac{K}{4\pi} \cdot min\{R_1 \cdot T_2, R_2 \cdot T_1\}.$$

Proof of Theorem 3.9. We recall that for trajectories of the vector field v the Gauss integral takes the form:

$$R_{abs}(\omega_1, \omega_2) = \frac{1}{4\pi} \int_{T_1} \int_{T_2} \frac{| < v_1 \wedge v_2, r_{12} > |}{\|r_{12}\|^3} \ dt_1 dt_2.$$

Here v_1, v_2 are the velocity vectors of ω_1, ω_2, respectively, (i.e. the values of the vector field v at the running points $p_1(t_1)$ and $p_2(t_2)$ on ω_1 and ω_2), and $r_{12} = p_2 - p_1$ is the vector joining the running points p_1 and p_2. So we have:

$$4\pi \cdot R_{abs}(\omega_1, \omega_2) = \int_{T_1} \int_{T_2} \frac{| < v_1 \wedge v_2, r_{12} > |}{\|r_{12}\|^3} \ dt_1 dt_2$$

$$= \int_{t_2 \in T_2} dt_2 \int_{t_2 \in T_1} \frac{| < v_1(t_1) \wedge (v_2(t_2) - v_1(t_1)), r_{12}(t_1, t_2) > |}{\|r_{12}\|^3} \ dt_1.$$

Indeed, the substraction of $v_1(t_1)$ from $v_2(t_2)$ does not change the triple product under the integral. Now since the vector field v is Lipschitz, we have: $\|v_2(t_2) - v_1(t_1)\| \leq K \cdot \|r_{12}\|$. Hence for the triple product we obtain:

$$| < v_1 \wedge (v_2 - v_1), r_{12} > | \leq \|\tilde{v}_1\| \cdot K \cdot \|r_{12}\| \cdot \|r_{12}\|,$$

where $\tilde{v}_1$ is the component of the vector v_1 orthogonal to the vector r_{12} (see Figure 6).

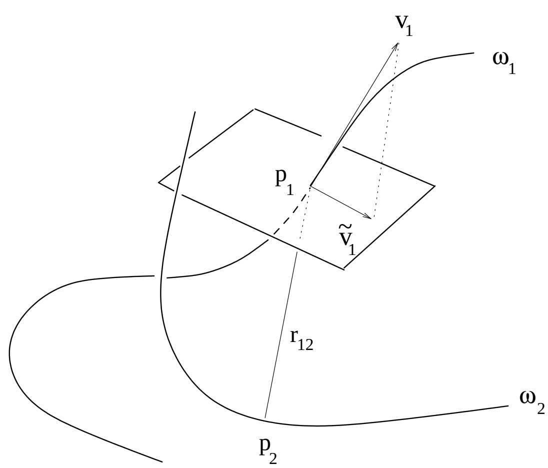

Figure 6

Therefore for the absolute rotation we get:

$$4\pi \cdot R_{abs}(\omega_1, \omega_2) \leq K \cdot \int_{t_2 \in T_2} dt_2 \int_{t_1 \in T_1} \frac{\|\tilde{v}_1(t_1, t_2)\|}{\|r_{12}(t_1, t_2)\|} \; dt_1.$$

In the interior integral the time t_2 and the corresponding point $p_2(t_2)$ on the curve ω_2 are fixed, while the integration runs over the curve ω_1, hence the integral $\int_{t_1 \in T_1} \frac{\|\tilde{v}_1(t_1, t_2)\|}{\|r_{12}(t_1, t_2)\|} \; dt_1$ is equal to the length of the spherical projection σ_1 of the curve ω_1 from the point $p_2(t_2)$, i.e. to $2\pi \cdot R_{abs}(\omega_1, p_2)$. By the assumptions, this rotation is uniformly in p_2 bounded by R_1. Hence the interior integral $\int_{t_1 \in T_1} \frac{\|\tilde{v}_1\|}{\|r_{12}\|} \; dt_1$ does not exceed R_1, and finally:

$$4\pi R_{abs}(\omega_1, \omega_2) \leq K \cdot T_2 \cdot R_1.$$

The setting of Theorem 3.9 is symmetric with respect to the trajectories ω_1 and ω_2. So interchanging these trajectories we get:

$$4\pi R_{abs}(\omega_1, \omega_2) \leq K \cdot T_1 \cdot R_2.$$

This completes the proof of Theorem 3.9. $\square$

Proof of Theorem 3.8. We use Theorem 3.4, which states that a rotation in time T of any trajectory of a Lipschitz vector field v in $\mathbb{R}^3$ around any point p does not exceed:

$$4 + K \cdot T.$$

Applying Corollary 3.10 we get:

$$R_{abs}(\omega_1, \omega_2) \leq \min\left(\frac{K}{\pi} \cdot T_1 + \frac{1}{4\pi} \cdot K^2 \cdot T_1 \cdot T_2, \frac{K}{\pi} \cdot T_2 + \frac{1}{4\pi} \cdot K^2 \cdot T_1 \cdot T_2, \right),$$

and Theorem 3.8 is proved. $\square$

Example of application: Logarithmic bound of the local rotation at singular points for analytic vector fields.

Let us consider the following situation. The vector field:

$$\frac{dv}{dt} = Lv + G(v)$$

is defined in a neighborhood of $O \in \mathbb{R}^3$ and has a non-degenerate linear part L with all the eigenvalues $\ell_j,\ j = 1, 2, 3$, having a negative real part: $\Re(\ell_j) \leq \ell < 0,\ j = 1, 2, 3$. In dynamical language, v has a non-degenerate sink at the origin (it is the case of our field in $\mathbb{R}^2$, in the Remark following Proposition 3.1). It is easy to see that the Lipschitz constant of v in a neighborhood U of the origin tends to the norm of L as U shrinks to the origin. The following theorem is an immediate corollary of the results of Section 3:

Theorem 3.10. — *For any two trajectories ω_1, ω_2 of the field v in a neighborhood of $O \in \mathbb{R}^3$, the absolute rotation $R_{abs}(\omega_1, \omega_2)$ grows at most logarithmically with the distance to the origin. More accurately, the rotation $R_{abs}(\omega_1, \omega_2, R, r)$ of the parts of ω_1, ω_2 between the spheres of the radii $R > r > 0$ satisfies:*

$$R_{abs}(\omega_1, \omega_2, R, r) \leq C\|L\| \frac{\log^2(R/r)}{\ell}.$$

Proof. This is a direct consequence of Theorem 3.8, since the Lipschitz constant of v in a neighborhood U of the origin tends to $\|L\|$ as U shrinks to the origin, while the time interval for both the trajectories between the spheres of the radii $R > r > 0$ is of order $\frac{\log(R/r)}{\ell}$. $\square$

Remark. Of course, one can easily show that the bound of Theorem 3.10 is sharp: consider a linear vector field:

$$\frac{dv}{dt} = Lv$$

in a neighborhood of $O \in \mathbb{R}^3$, with a non-degenerate linear part L having all its eigenvalues $\ell_j,\ j = 1, 2, 3$, with negative real part: $Re(\ell_j) \leq \ell < 0,\ j = 1, 2, 3$. Assume in addition that $\ell_1 \in \mathbb{R}$, while ℓ_2 and ℓ_3 are conjugate: $\ell_{2,3} = \alpha \pm i\beta$. Then the solutions are $x_1 = C_1 \cdot \exp(\ell_1 t),\ x_2 = C_2 \cdot \exp(\alpha t) \cdot \sin(\beta t),\ \ x_3 = C_3 \cdot \exp(\alpha t) \cdot \cos(\beta t)$, and the trajectories rotate around the x_1-axis and one around another exactly as prescribed by the upper bound given in Theorem 3.8.

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